OGLEDNI KOLOKVIJ – MATEMATIKA 1

1. (i) $z = \frac{3}{4} + \frac{3\sqrt{3}}{4} i$

$$|z| = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{3\sqrt{3}}{4}\right)^2} = \sqrt{\left(\frac{3}{4}\right)^2 (1+3)} = \frac{3}{2}$$

$$\cos\alpha = \frac{3/4}{3/2} = \frac{1}{2}, \quad \sin\alpha = \frac{3\sqrt{3}/4}{3/2} = \frac{\sqrt{3}}{2} \implies \alpha = 60^\circ$$

$$\implies \boxed{z = \frac{3}{2}\left(\cos 60^\circ + i \sin 60^\circ\right)} \qquad (1 \text{ bod})$$

(ii)

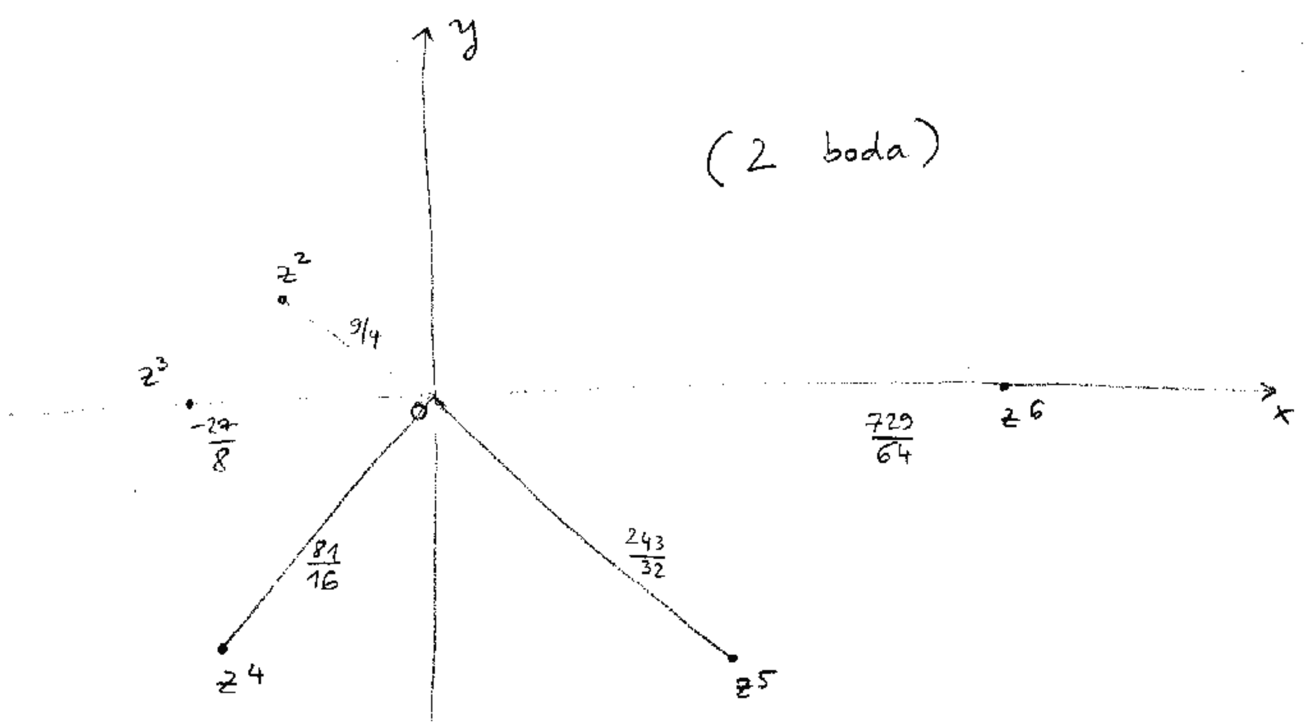
$$z^2 = \left(\frac{3}{2}\right)^2 (\cos 120^\circ + i \sin 120^\circ) = \frac{9}{4}(\cos 120^\circ + i \sin 120^\circ) = \frac{-9}{8} + \frac{9\sqrt{3}}{8} i$$

$$z^3 = \left(\frac{3}{2}\right)^3 (\cos 180^\circ + i \sin 180^\circ) = -\frac{27}{8}$$

$$z^4 = \left(\frac{3}{2}\right)^4 (\cos 240^\circ + i \sin 240^\circ) = \frac{81}{16}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = \frac{-81}{32} - \frac{81\sqrt{3}}{32} i$$

$$z^5 = \left(\frac{3}{2}\right)^5 (\cos 300^\circ + i \sin 300^\circ) = \frac{243}{32}\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = \frac{243}{64} - \frac{243\sqrt{3}}{64} i$$

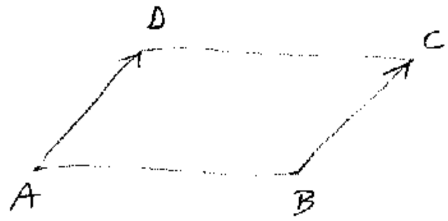
$$z^6 = \left(\frac{3}{2}\right)^6 (\cos 360^\circ + i \sin 360^\circ) = \frac{729}{64}$$



(2 boda)

2. i) $A(2,3,4)$, $B(3,-1,0)$, $C(1,1,1)$

$D(x,y,z)$



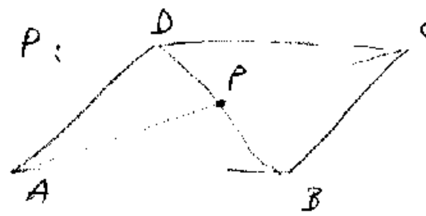
$\vec{AD} = \vec{BC}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} \Rightarrow \boxed{D(0,5,5)} \quad (1 \text{ bod})$$

ii) Dijagonale se sijeku u polovištu P:



$\vec{AP} = \frac{1}{2}\vec{AC}$; $P(x,y,z)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right) = \frac{1}{2}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 2 \\ 5/2 \end{bmatrix} \Rightarrow P\left(\frac{3}{2}, 2, \frac{5}{2}\right)$$

$$\boxed{\vec{r}_P = \frac{3}{2}\vec{i} + 2\vec{j} + \frac{5}{2}\vec{k}} \quad (1 \text{ bod})$$

iii) Napišimo matricu simetrije obzirom na xy-ravninu:

$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ i primijenimo je na sve 4 točke-vrhove paralelograma:

A': $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \Rightarrow A'(2,3,-4)$

B': $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \Rightarrow B'(3,-1,0)$

C': $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow C'(1,1,-1)$

D': $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix} \Rightarrow D'(0,5,-5)$

$\Rightarrow$ To je paralelogram $A'B'C'D'$, gdje su $A'(2,3,-4)$, $B'(3,-1,0)$, $C'(1,1,-1)$ i $D'(0,5,-5)$

(1 bod)

3. i)

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \underset{\mathrm{III} \to \mathrm{I} - \mathrm{III}}{=} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} \underset{\substack{\mathrm{I} \to \mathrm{II} \cdot (-1) + \mathrm{I} \\ \mathrm{III} \to \mathrm{II} \cdot 1 + \mathrm{III}}}{=} \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} \underset{\substack{\mathrm{I} \to \mathrm{III} \cdot 2 + \mathrm{I} \\ \mathrm{II} \to \mathrm{III} \cdot (-2) + \mathrm{II}}}{=} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 1 \cdot 1 \cdot 2 = 2$$

(1 bod)

ii)

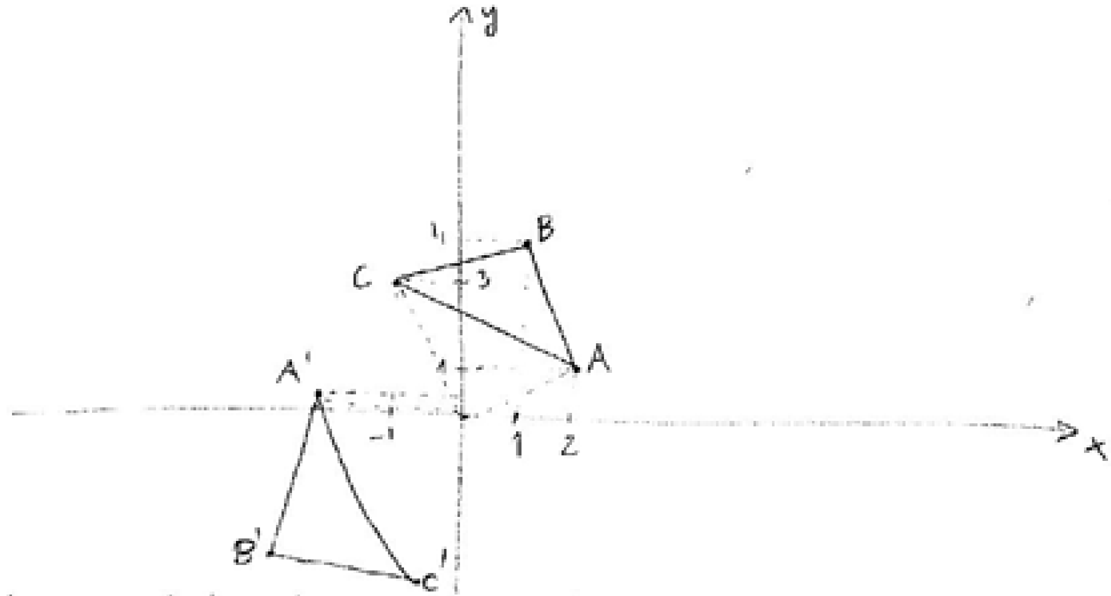
$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\mathrm{III} \to \mathrm{I} \cdot (-1) + \mathrm{III}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \underset{\substack{\mathrm{I} \to \mathrm{II} \cdot (-1) + \mathrm{I} \\ \mathrm{III} \to \mathrm{II} \cdot 1 + \mathrm{III}}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \underset{\mathrm{III} \to \mathrm{III} \cdot \frac{1}{2}}{\sim}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right] \underset{\substack{\mathrm{I} \to \mathrm{III} \cdot 1 + \mathrm{I} \\ \mathrm{II} \to \mathrm{III} \cdot (-1) + \mathrm{II}}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} = \underline{\frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}}$$

(2 boda)

4. (i) (1 bod)



(ii) Rotacija za kut $\alpha = 150°$ ima matricu

$$S_\alpha = \begin{bmatrix} \cos 150° & -\sin 150° \\ \sin 150° & \cos 150° \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$$

Sada svaki vrh trokuta rotiramo za $150°$:

$$A' = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} - 1/2 \\ 1 - \sqrt{3}/2 \end{bmatrix} \qquad -2.23,\ 0.13$$

$$B' = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 - 2 \\ 1/2 - 2\sqrt{3} \end{bmatrix} \qquad -2.86,\ -2.96$$

$$C' = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 - 3/2 \\ -1/2 - 3\sqrt{3}/2 \end{bmatrix} \qquad -0.63,\ -3.09$$

Rotacijom se dobije trokut $A'B'C'$, gdje su:

$A'\left(-\sqrt{3} - \frac{1}{2}, 1 - \sqrt{3}/2\right)$, $B'\left(-2 - \frac{\sqrt{3}}{2}, \frac{1}{2} - 2\sqrt{3}\right)$, $C'\left(\frac{\sqrt{3}}{2} - \frac{3}{2}, -\frac{1}{2} - \frac{3\sqrt{3}}{2}\right)$

(2 boda)

5. (i)

$$x+y=3$$
$$y+z=5$$
$$x+z=4$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 5 \\ 1 & 0 & 1 & 4 \end{array}\right] \underset{III \to I\cdot(-1)+III}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 5 \\ 0 & -1 & 1 & 1 \end{array}\right] \underset{\substack{I \to II\cdot(-1)+I \\ III \to II\cdot 1 + III}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 \end{array}\right] /:2 \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \underset{\substack{I \to III\, 1+I \\ II \to III\cdot(-1)+II}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\Rightarrow$ $x=1$, $y=2$, $z=3$ (1 bod) sustav ima jedinstveno rješenje

(ii)

$$x+y=3$$
$$y+z=5$$
$$2x+y-z=1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 5 \\ 2 & 1 & -1 & 1 \end{array}\right] \underset{III \to I\cdot(-2)+III}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 5 \\ 0 & -1 & -1 & -5 \end{array}\right] \underset{\substack{I \to II\cdot(-1)+I \\ III \to II\cdot 1+III}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 5 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$$

$$x-z=-2$$
$$y+z=5$$

Uzmimo $z=t, t\in\mathbb{R} \Rightarrow$ $x=-2+t$, $y=5-t$

$\rightarrow$ Sustav ima beskonačno mnogo rješenja dana s:

$$\left.\begin{array}{l} x=t-2 \\ y=5-t \\ z=t \\ t\in\mathbb{R} \end{array}\right\} \quad (2 \text{ boda})$$